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On rings satisfying the identity  $(X - X^n)^2 = 0$

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## ON RINGS SATISFYING THE IDENTITY $(X - X^n)^2 = 0$

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Throughout,  $R$  will represent a ring with center  $C$ , and  $N$  the set of nilpotent elements in  $R$ . Let  $n$  be a positive integer greater than 1, and  $E_n$  the set of elements  $x$  in  $R$  such that  $x = x^n$ .

We consider the following properties:

- (i)  $N$  is commutative.
- (ii) $_n^*$   $(x - x^n)(y - y^n) = 0$  for all  $x, y \in R$ .
- (ii) $_n$   $(x - x^n)^2 = 0$  for all  $x \in R$ .
- (ii) $'_n$   $(x - x^n)^n = 0$  for all  $x \in R$ .
- (iii) $_n$  Any  $x \in R$  may be written in at most one way in the form  $x = b + a$ , where  $b \in E_n$  and  $a \in N$ . (There may be elements  $x$  in  $R$  which cannot be written in the given form.)

If  $R$  satisfies (ii) $_n^*$  and (iii) $_n$ , then  $R$  is called a *generalized  $n$ -ring*. Following [4],  $R$  is called a *generalized  $n$ -like ring* if  $(xy)^n - xy^n - x^n y + xy = 0$  for all  $x, y \in R$ , or equivalently, if  $(x - x^n)(y - y^n) = 0$  and  $(xy)^n = x^n y^n$  for all  $x, y \in R$  (see [4, Lemma 3]).

The major purpose of this paper is to prove the following

**Theorem 1.** *If  $R$  satisfies (i), (ii) $_n$  and (iii) $_n$ , then  $R$  is commutative.*

In preparation for proving Theorem 1, we state the next lemma.

**Lemma 1.** (1) *Let  $R$  be a ring satisfying (i) and (ii) $_n$ . Then  $N$  is a commutative nil ideal of bounded index at most 2. If there exists an integer  $m > 1$  such that  $m^2 x^4 = 0$  for all  $x \in R$ , then  $x^{m^2} \in E_n$  for all  $x \in R$ .*

(2) *If  $R$  satisfies (i) and (ii) $_n$ , then there exists a finite set  $P$  of prime numbers such that  $R = \sum_{p \in P} R^{(p)}$ , where  $R^{(p)} = \{x \in R \mid px \in N\}$ .*

(3) *Let  $R$  be a ring satisfying (i), (ii) $_n$  and (iii) $_n$ . If there exists an integer  $m > 1$  such that  $m^2 x^4 = 0$  for all  $x \in R$ , then  $[x^{m^2}, a] = 0$  for all  $x \in R$  and  $a \in N$ .*

*Proof.* (1) By (ii) $_n$ , there holds  $x^{2n} = 2x^{n+1} - x^2$ . Hence,  $N$  is a commutative nil ideal of bounded index at most 2 by [2, Lemma 2 (2)]. Furthermore, an easy induction shows that  $x^{\mu n - \mu + 2} = \mu x^{n+1} - (\mu - 1)x^2$ , and so  $x^{\mu n} = \mu(x^{n+\mu-1} - x^\mu) + x^\mu$  for any positive integer  $\mu$ ; in particular,  $x^{m^2 n} =$

$$m^2(x^{n+m^2-1}-x^{m^2})+x^{m^2}=x^{m^2}.$$

(2) Let  $m=(2^n-2)^2$ . Since  $N$  is an ideal of  $R$  by (1), we see that  $(2^n-2)x=2^n(x-x^n)-\{2x-(2x)^n\} \in N$  for all  $x \in R$ , i.e.,  $m(R/N)=0$ . As is well known, the factor ring  $R/N$  satisfying the polynomial identity  $X-X^n=0$  is a subdirect sum of finite fields (see, e.g., [1, Theorem 19]). Noting here that  $m(R/N)=0$ , we can easily see the assertion. Needless to say, every  $R^{(p)}$  is an ideal of  $R$  containing  $N$ .

(3) Let  $x \in R$ , and  $a \in N$ . According to (1), we have  $(x+a)^{m^2}=x^{m^2}+a'+a''$ , where  $x^{m^2} \in E_n$ ,  $a'=\sum_{i=0}^{m^2-1}x^{m^2-i-1}ax^i \in N$  and  $a'' \in N^2 \subseteq C$ . Since  $(x+a)^{m^2}$  is also in  $E_n$ , (iii)<sub>n</sub> shows that  $a'+a''=0$ . Hence,  $[x^{m^2},a]=[x,a']=[x,a'']+[x,a'']=0$ .

*Proof of Theorem 1.* In view of Lemma 1 (2), there exists a finite set  $P$  of prime numbers such that  $R=\sum_{p \in P} R^{(p)}$ , where  $R^{(p)}$  is the ideal of  $R$  containing  $N$  defined by  $\{x \in R \mid px \in N\}$ . Obviously, (i), (ii)<sub>n</sub> and (iii)<sub>n</sub> are inherited by the ideal  $R^{(p)}$ . Since  $N$  is a nil ideal of bounded index at most 2 (Lemma 1 (1)), we see that  $p^2x^2=0$  for all  $x \in R^{(p)}$ , and so  $[x^{p^4},a]=0$  for all  $x \in R^{(p)}$  and  $a \in N$  (Lemma 1 (3)). As is well known, the factor ring  $R^{(p)}/N$  satisfying the polynomial identity  $X-X^n=0$  is a subdirect sum of finite fields of characteristic  $p$ , and hence we can find a positive integer  $k$  such that  $x^{p^k}-x \in N$  for all  $x \in R^{(p)}$ . Now, let  $x \in R^{(p)}$  and  $a \in N$ . Since  $[x^{p^4},a]=0$  and  $x^{p^{4k}}-x \in N$ , we get  $[x,a]=0$  by (i), which shows that  $N$  is in the center of  $R^{(p)}$ . Hence,  $N$  is contained in the center of  $R$ , and therefore  $R$  is commutative by [1, Theorem 19].

If  $R$  is a generalized  $n$ -ring, it is easy to see that  $N^2=0$ , and so  $N$  is commutative. Thus, as a direct consequence of Theorem 1, we have

**Corollary 1.** *Every generalized  $n$ -ring is commutative. In particular, every generalized  $n$ -like ring satisfying (iii)<sub>n</sub> is commutative.*

**Corollary 2.** *Suppose that there exists an integer  $m > 1$  such that  $(m, n-1)=1$  and  $mN=0$ . Suppose that  $R$  satisfies (i) and (ii)<sub>n</sub>. Then,  $R$  is commutative if and only if  $R$  satisfies (iii)<sub>n</sub>.*

*Proof.* In view of Theorem 1, it suffices to show that if  $R$  is commutative then (iii)<sub>n</sub> is satisfied. Suppose that both  $b$  and  $b+a$  are in  $E_n$  with some  $a \in N$ . Then  $b+nab^{n-1}=(b+a)^n=b+a$  (Lemma 1 (1)), and so  $nab^{n-1}=a$ , whence it follows that  $nab=nab^n=ab$ . Hence,  $na=n^2ab^{n-1}=nab^{n-1}=a$ , namely  $(n-1)a=0$ . Since  $ma=0$  and  $(m, n-1)=1$ , we get  $a=0$ , proving (iii)<sub>n</sub>.

Next, motivated by [3, Theorem 1], we prove the following

**Theorem 2.** *Let  $p$  be a prime. If  $R$  satisfies (i),  $(ii)_p$  and  $pR=0$ , then the following are equivalent :*

- 1)  $R$  is commutative.
- 2)  $R$  satisfies  $(iii)_p$ .
- 3)  $E_p$  is a subring of  $R$ .
- 4)  $E_p$  is an additive subgroup of  $R$ .
- 5)  $E_p$  is central.

*Proof.* Obviously,  $x^p \in E_p$  for any  $x \in R$ , and  $N$  is a commutative nil ideal of bounded index at most  $p$  by [2, Lemma 2 (2)]. Then, it is easy to see that  $1) \Rightarrow 3) \Rightarrow 4) \Rightarrow 2)$  and  $1) \Leftrightarrow 5)$ .

$2) \Rightarrow 1)$ . Let  $x \in R$ , and  $a \in N$ . Then we have  $(x+a)^p = x^p + a' + a''$ , where  $x^p \in E_p$ ,  $a' = \sum_{i=0}^{p-1} x^{p-i-1} ax^i \in N$  and  $a'' \in N^2 \subseteq C$ . Since  $(x+a)^p$  is also in  $E_p$ ,  $(iii)_p$  shows that  $a' + a'' = 0$ . Hence,  $[x^p, a] = [x, a'] = [x, a'] + [x, a''] = 0$ , and therefore  $[x, a] = [x^p, a] + [x - x^p, a] = 0$ , which shows that  $N \subseteq C$ . Now,  $R$  is commutative by [1, Theorem 19].

**Examples.** (1) The commutative ring  $R = \mathbb{Z}/4\mathbb{Z}$  satisfies  $(ii)_3^*$ , but does not  $(iii)_3$ ; the commutative ring  $\mathbb{Z}/8\mathbb{Z}$  satisfies  $(ii)_3$ , but does neither  $(ii)_3$  nor  $(iii)_3$ .

(2) Let  $p$  be a prime. Then  $R = \left\{ \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} \mid a, b \in \text{GF}(p) \right\}$  is a non-commutative ring satisfying  $(ii)_p^*$  and  $pR=0$ .

(3) Let  $R = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & b \\ 0 & 0 & a \end{pmatrix} \mid a, b, c \in \text{GF}(3) \right\}$ . Then  $R$  is a commutative ring satisfying  $(ii)_3$  and  $3R=0$ , but not  $(ii)_3^*$ .

(4) Let  $R = \left\{ \begin{pmatrix} a & b & c & d \\ 0 & a & 0 & c \\ 0 & 0 & a & b \\ 0 & 0 & 0 & a \end{pmatrix} \mid a, b, c, d \in \text{GF}(2) \right\}$ . Then  $R$  is a commutative ring satisfying  $(ii)_2 = (ii)_2^*$  and  $2R=0$ , but not  $(ii)_2^*$ .

These examples give the following table, where (c) signifies the property that  $R$  is commutative.

$$\begin{array}{c}
 (ii)_n^* \wedge (c) \xleftrightarrow{\quad} (ii)_n^* \wedge (iii)_n \\
 \updownarrow \quad \quad \quad \updownarrow \\
 (ii)_n^* \xleftrightarrow{\quad} (i) \wedge (ii)_n \wedge (iii)_n \iff (ii)_n \wedge (iii)_n \wedge (c) \\
 \quad \quad \quad \updownarrow \quad \quad \quad \updownarrow \\
 \quad \quad \quad (ii)_n \wedge (c) \quad \quad \quad (ii)_n' \wedge (iii)_n \wedge (c) \\
 \quad \quad \quad \updownarrow \quad \quad \quad \updownarrow \\
 \quad \quad \quad (ii)_n' \wedge (c) \quad \quad \quad (ii)_n' \wedge (iii)_n \wedge (c)
 \end{array}$$

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